J. Fluid Mech. (1979), *vol.* 92, *part* 4, *pp.* 751–766
Printed in Great Britain

Three-dimensional natural convection in a confined porous medium heated from below

By ROLAND N. HORNE
Department of Petroleum Engineering, Stanford University, California 94305†

(Received 30 June 1978 and in revised form 18 October 1978)

Previous analyses of natural convection in a porous medium have drawn seemingly contradictory conclusions as to whether the motion is two- or three-dimensional. This investigation uses numerical results to show the relationship between previous contending observations, and demonstrates that there exists more than one mode of convection for any particular physical configuration and Rayleigh number. In some cases, a particular flow situation may be stable even though it does not maximize the energy transfer across the system.

The methods used are based on the efficient numerical solution of the governing equations, formulated with the definition of a vector potential. This approach is shown to be superior to formulating the equations in terms of pressure.

For a cubic region the flow pattern at a particular value of the Rayleigh number is not unique and is determined by the initial conditions. In some cases there exist four alternatives, two- and three-dimensional, steady and unsteady.

1. Introduction

Although the flow of fluid through porous media by the process of natural convection has received considerable attention since the earliest work by Horton & Rogers (1945) there still exist major points of interest and contention. Early work (for example, Lapwood 1948; and Katto & Masuoka 1967) investigated the critical conditions (Rayleigh number) at which the heat transport process changes from purely conductive to convective transfer, and later studies (for example Elder 1967) concentrated on the convective flows for conditions moderately above this critical Rayleigh number R (which is $4\pi^2$ for an infinitely wide layer). Combarnous & LeFur (1969) observed in their experiments that another modal transition occurred at a second critical Rayleigh number R_2 (which they found to be within the range 240–280), and the new convective regime above this Rayleigh number was described by Caltagirone, Cloupeau & Combarnous (1971) to be permanently unsteady, with motion occurring in fluctuating two-dimensional convective rolls. These fluctuating states were also observed experimentally and numerically by Horne & O'Sullivan (1974) and numerically by Caltagirone (1974, 1975). Straus (1974), in an analytical examination of the stability of steady two-dimensional rolls to three-dimensional disturbances, concluded that no two-dimensional convective pattern is stable above the second critical Rayleigh number

† Current address: Department of Theoretical and Applied Mechanics, University of Auckland, New Zealand.

0022-1120/79/4254-6840 $02.00

R_2 which he showed to be dependent on the aspect ratio of the roll but in no case larger than 380. This work however was done for the infinitely wide, unconfined layer, and bears only indirectly on finite domain applications.

Although the results of Straus (1974) show that a two-dimensional roll pattern is stable inside an envelope in the Rayleigh number/roll wavenumber plane, it has not as yet been conclusively determined what kinds of flow exist outside of this envelope or in confined systems. The experiments of Caltagirone *et al.* (1971) performed in a narrow three-dimensional space (38 cm long, 2 cm wide and 4–6 cm deep), suggest the two-dimensional fluctuating convective state, a fact which led to the various two-dimensional numerical studies of this state. However, Horne & O'Sullivan (1974) have demonstrated that the flow is not unique at Rayleigh numbers up to about twice R_2 and that there exists both steady and unsteady alternatives. A similar situation was observed by Caltagirone (1974, 1975) who obtained both steady and unsteady flows at the same Rayleigh number by altering the aspect ratio of the convective cells. The reason for the existence of these alternatives is indicated by Straus' (1974) stability envelope, at a particular Rayleigh number within a certain range, convective cells of certain wave numbers lie within the envelope of stable steady two-dimensional flows while others lie outside it, although as pointed out by Straus & Schubert (1978) both the base state and the disturbance must fit between the boundaries for this envelope to have meaning for a finite system. However since these studies were all explicitly two-dimensional the results obtained represent only a subset of possible three-dimensional flows; the two-dimensional flows observed may well be stable and viable forms (as is clear from the two-dimensional fluctuating experimental flows), nevertheless the nature of any precluded three-dimensional flow is not understood. An extension of these analyses to three dimensions is required.

Experimental results reported by Bories, Combarnous & Jaffrenou (1972) in a wide layer ($46 \times 66 \times 5\frac{1}{2}$ cm deep) show three-dimensional flows moving in steady polyhedral convective cells at Rayleigh numbers between the first and second critical values (R_1 and R_2). Such flows clearly lie outside Straus' (1974) stability envelope for the infinite layer. Holst & Aziz (1972), in a transient three-dimensional numerical analysis of the evolution of convective patterns in cubic boxes, observed that while at low Rayleigh numbers (close to R_1) two-dimensional rolls transfer more energy (indicated by the Nusselt number) than three-dimensional flows, at higher Rayleigh numbers (but still between R_1 and R_2) the reverse is true. The first result was confirmed by Zebib & Kassoy (1978). However, counter to the idea of Platzman (1965), Horne & O'Sullivan (1974) have already demonstrated cases in which the energy transfer is not maximized (even though the tendency may exist) and it is anticipated that many stable flows will exist for any particular physical configuration and Rayleigh number. Beck (1972) has illustrated the many possible convective regimes at conditions even only marginally above the first critical Rayleigh number.

In addition to the possibilities of separate flow regimes, the interaction between them must be considered. Zebib & Kassoy (1978) used the method of weakly non-linear analysis (Palm, Weber & Kvernvold 1972) to develop a two-term expansion for the temperature and velocity fields and for the Nusselt number at values of the Rayleigh number slightly above R_1. Their results show that the Nusselt numbers associated with either of the possible two-dimensional roll motions (aligned in either horizontal co-ordinate direction) are identical (i.e. either orientation is equally probable), and are larger than the Nusselt number of the three-dimensional motion resulting from the non-linear interaction of these two motions. This could be an indication of transformability of a particular flow regime; the three-dimensional flow would increase its energy transfer by transforming into either of the two possible two-dimensional patterns (since either is equally probable presumably some perturbation in favour of one would be required). The reverse transformation would supposedly be less likely. It should be noted however that the 'three-dimensional' motion considered by Zebib & Kassoy was a superposition of orthogonal rolls ($\sin \alpha x + \sin \beta y$) rather than an intrinsically three-dimensional motion ($\sin \alpha x \sin \beta y$).

Recently, Straus & Schubert (1978) have specifically examined the stability of convective flow in a finite three-dimensional box, and have isolated the particular box dimensions for which *no* stable two-dimensional flow exists. Although they considered confining boxes that were taller than wide, this work is particularly significant in that it has determined for the first time conditions under which a particular flow can *not* exist, although the tall boxes they considered admittedly do not allow the existence of very many disturbances. It is clear that proof that a particular flow regime *can* exist is not very appropriate to nonlinearly unstable flows for which it has been shown that more than one regime can exist under identical conditions. The work of Straus & Schubert (1978) does not directly shed light on the question of whether three-dimensional convective states and two-dimensional fluctuating convective states are actually alternatives to one another or if they exist under different conditions. However in a still more recent work, Schubert & Straus (1978) have shown that transitions from steady to oscillatory two-dimensional convection cannot occur for the unconfined system since the steady two-dimensional flow becomes unstable to three-dimensional disturbances before becoming unstable in a strictly two-dimensional way by becoming oscillatory. On the other hand, in the confined system it is necessary that the three-dimensional disturbance fits into the box; if it does not, then the flow may well undergo the transition to oscillatory two-dimensional convection first.

With the object of scrutinizing these various points, this investigation extends the work of Holst & Aziz (1972), making use of more sophisticated numerical techniques which permit calculation of flow patterns at higher Rayleigh numbers, on finer finite difference meshes and for much longer periods of time. Initially, as a test on the procedure the flow situations considered by Zebib & Kassoy (1978) are confirmed for small Rayleigh numbers just greater than R_1; subsequently, the stability of three-dimensional flows at Rayleigh numbers above R_2 is evaluated in the light of previous two-dimensional studies (Horne & O'Sullivan, 1974; Caltagirone, 1974, 1975).

2. Description of the problem

Consider the natural convective motion of a fluid filling a homogeneous, isotropic porous medium confined on all sides by an impermeable rectangular box. The box is heated from below, cooled from above, and insulated on all vertical sides. The equations of motion governing the flow of fluid in this situation are the conservation of mass, momentum and energy,

$$\nabla . \mathbf{U} = 0, \tag{1}$$

$$\mathbf{U} = \frac{R}{\lambda}(\mathbf{j}\theta) - \nabla P, \tag{2}$$

and
$$\frac{\partial \theta}{\partial \tau} = \nabla^2\theta - \lambda \mathbf{U}\frac{\partial \theta}{\partial \mathbf{X}}, \tag{3}$$

where $\mathbf{U}$, P and θ are the non-dimensional velocity, pressure and temperature fields, $\mathbf{X}$ and τ are the non-dimensional space vector and time, $\mathbf{j}$ is a unit normal vector pointing vertically upward and λ is the ratio of the volumetric heat capacity of the fluid to that of the saturated formation. The Rayleigh number R is defined by

$$R = \frac{gak\Delta T\alpha\lambda}{\kappa\nu}. \tag{4}$$

The physical parameters governing the problem are g the acceleration due to gravity, a the depth of the flow region, k the permeability of the medium, ΔT the temperature differential between the heat source and sink, α the volumetric thermal expansion coefficient of the saturating fluid, ν its kinematic viscosity, and κ the thermal diffusivity of the fluid-filled medium. The non-dimensional forms of the dependent and independent variables are given by

$$\tau = (\kappa/a^2)\,t \quad \text{(time)},$$
$$\mathbf{X} = \mathbf{x}/a \quad \text{(space)},$$
$$\mathbf{U} = (a/\kappa)\,\mathbf{u} \quad \text{(velocity)},$$
$$P = \frac{k}{\rho\kappa\nu}p \quad \text{(pressure)},$$
$$\theta = (T - T_{\min})/\Delta T \quad \text{(temperature)},$$

where $T_{\min}$ is the minimum boundary temperature in the region and lower case characters represent variables equivalent to the dimensionless variable being defined.

Inherent in the derivation of these equations are a number of assumptions:

(*a*) The Boussinesq assumption (see Yih 1969, p. 441; also Straus & Schubert 1977).

(*b*) Inertial effects are small (low Reynolds number; see Batchelor 1967, p. 223).

(*c*) Viscosity ν is constant (see Straus & Schubert 1977).

(*d*) Thermal dispersion is negligible (see Rubin 1974).

(*e*) Saturating fluid and porous solid are in thermal equilibrium (see Combarnous 1972).

Each of these assumptions incur certain small inaccuracies, and the references cited give discussions of the conditions under which these inaccuracies become significant. For the present study none is of major importance.

The governing equations may be made more tractable in either of two alternative formulations, one involving only pressure and temperature and the other involving temperature and a vector potential function.

The *pressure formulation* is obtained by eliminating velocity between equations (1) and (2), after taking the divergence of (2) and substituting (1), thus

$$\nabla^2 P = \partial\theta/\partial Y, \tag{5}$$

where Y is the component of $\mathbf{X}$ pointing vertically upward, and P has been scaled by a factor R/λ. The Laplacian operator is defined

$$\nabla^2 \equiv \partial/\partial X + \partial/\partial Y + \partial/\partial Z.$$

The velocity vector may then also be eliminated between equations (2) and (3), such that

$$\frac{\partial \theta}{\partial \tau} = \nabla^2 \theta + R\frac{\partial P}{\partial \mathbf{X}} \cdot \frac{\partial \theta}{\partial \mathbf{X}} - R\theta \frac{\partial \theta}{\partial Y}, \tag{6}$$

and equations (5) and (6) now form the governing pair for the problem.

The boundary conditions on θ and P for the problem region described earlier are:

$$\partial\theta/\partial\mathbf{n} = 0 \quad \text{on all vertical boundary surfaces,}$$
$$\theta = 0 \quad \text{on} \quad Y = 0$$
$$\theta = 1 \quad \text{on} \quad Y = 1,$$

and

$$\partial P/\partial \mathbf{n} = 0 \quad \text{on all boundary surfaces,}$$

where $\mathbf{n}$ is the outward pointing normal vector.

Due to the solenoidal form of $\mathbf{U}$ in equation (1), there is an alternative formulation described by Holst & Aziz (1972) in which a vector potential $\boldsymbol{\varphi}$ is introduced such that

$$\frac{\lambda}{R}\mathbf{U} = \nabla \times \boldsymbol{\varphi}. \tag{7}$$

Then, taking the curl of equation (2),

$$\frac{\lambda}{R}(\nabla \times \mathbf{U}) = \nabla \times (\mathbf{j}\theta). \tag{8}$$

Substituting the definition (7) of $\boldsymbol{\varphi}$ and making use of the fact that it is arbitrary to the gradient of a scalar (hence its divergence may be set to zero),

$$\nabla^2 \boldsymbol{\varphi} = (\partial\theta/\partial Z, 0, -\partial\theta/\partial X). \tag{9}$$

Considering the boundary conditions on $\boldsymbol{\varphi}$ for a closed, impermeable, cubic box, it is seen that

$$\partial\phi_1/\partial X = \phi_2 = \phi_3 = 0 \quad \text{on} \quad X = 0, 1,$$
$$\partial\phi_2/\partial Y = \phi_1 = \phi_3 = 0 \quad \text{on} \quad Y = 0, 1,$$

and

$$\partial\phi_3/\partial Z = \phi_1 = \phi_2 = 0 \quad \text{on} \quad Z = 0, 1.$$

As a consequence of these boundary conditions and equation (9), the component ϕ_2 of the vector potential is zero everywhere within the region. Thus the final form of the vector potential equations may be written as

$$\nabla^2\phi_1 = \partial\theta/\partial Z, \quad \nabla^2\phi_3 = \partial\theta/\partial X, \tag{10), (11}$$

together with the energy equation

$$\frac{\partial \theta}{\partial \tau} = \nabla^2\theta - R\left[\frac{\partial(\theta, \phi_3)}{\partial(X, Y)} + \frac{\partial(\theta, \phi_1)}{\partial(Y, Z)}\right], \tag{12}$$

where the Jacobian operation is described typically by

$$\frac{\partial(\theta, \phi)}{\partial(X, Y)} \equiv \frac{\partial \theta}{\partial X}\frac{\partial \phi}{\partial Y} - \frac{\partial \theta}{\partial Y}\frac{\partial \phi}{\partial X}. \tag{13}$$

Even though it may seem that the pressure formulation with its two equations is more manageable than the vector potential formulation which has three, in fact the vector potential equations may be solved numerically much faster and with greater accuracy.

3. Numerical solution

The numerical solution of nonlinear partial differential equations is seldom tractable in more than two dimensions, however the forms of the governing equations and the regular boundary conditions in this case permit specialized techniques to be applied, allowing accurate, long time solutions at reasonable expense. As in the two-dimensional formulation of this problem described by Horne & O'Sullivan (1974), the finite difference solution depends firstly on the rapid direct solution of the Poisson equation [(5) or (10) and (11)] and secondly on the accurate, energy conservative differencing of the advection terms in the energy equation [(6) or (12)].

The direct, non-iterative, odd–even reduction algorithm for the solution of the Poisson equation (as described by Busbee, Golub & Nielson 1970) can be extended to applications on three-dimensional finite difference meshes. In order to evaluate which of the alternative formulations of the governing equations would be most suitable, it was necessary to develop two forms of the algorithm corresponding to the different types of boundary conditions on pressure and vector potential. Calculation of a pressure field on a $17 \times 17 \times 17$ grid requires representation on all 4913 modes of the mesh, since all six boundary surfaces have Neumann type boundary conditions. On the other hand, calculation of either vector potential field requires representation on only $15 \times 15 \times 17$ or 3825 modes of the mesh since on four of the six boundary surfaces the function is already specified by the Dirichlet type boundary condition. In addition the algorithm for all Neumann boundaries incurs much greater programming overhead. Thus overall the numerical solution of the single equation (5) actually requires about 30 % more CPU time than the solutions to *both* (10) and (11) in the vector potential formulation.

The energy equation has been successfully solved in Horne & O'Sullivan (1974) using an explicit, forward time stepping procedure. However the development of solutions over a long period of time was critically dependent on the representation of the advection terms [terms like $(\partial\phi/\partial Y)(\partial\theta/\partial X)$ for example] without accumulating false, numerically generated energy. This artificial energy may be dissipated to some extent by the thermal dispersion inherent in the equation, but only at relatively small Rayleigh number (say as high as 100). The difference scheme derived by Arakawa (1966) specifically conserves kinetic energy and mean square temperature in an application such as this (in the absence of time differencing errors), however unfortunately it is applicable only to combinations of first derivatives in Jacobian form. During the course of this investigation several attempts were made to derive an equivalent conservative scheme to represent the non-Jacobian advection terms which appear in the pressure-formulated energy equation (6), however without success. There exists an alternative and very powerful finite difference scheme for evaluating first derivatives that is due to Kreiss and is described by Orszag & Israeli (1974) and Hirsh (1975). Kreiss' scheme was tested against Arakawa's during the investigation and gave identical results for a two-dimensional problem at high Rayleigh number, however, it has not been proven to conserve (or more precisely semi-conserve) energy in the same way. Furthermore the method is implicit and was more expensive (by a factor of about 1·5) to compute.

Thus overall a solution procedure for the coupled energy and momentum equations is faster for the vector potential formulation by a factor of about 1·4, for comparable

4th-order accuracy in the advection terms. Since the pressure formulation also could not specifically be shown to avoid artificial generation of numerical energy, the vector potential formulation was selected as the most suitable approach.

Run times for solutions on a $17 \times 17 \times 17$ mesh (used for the cubic regions) were 1·5 s/time step on an IBM 370/168 machine. A typical run for about 500 time steps consumed on the order of $150 of computer time. The time step used in the calculations varied in size between 0·0005 for Rayleigh numbers of 75 to 0·00025 for Rayleigh numbers of 300.

The choice of mesh size is most severely restricted by financial limitations, and the $17 \times 17 \times 17$ grid is admittedly coarse for Rayleigh numbers in excess of 400, see Horne & O'Sullivan (1974). However the moderate values of R, the method can be used with reasonable confidence, although it is a property of finite difference methods that the Nusselt number tends to be over-estimated owing to the necessity of taking one-sided differences at the boundary. It should be remembered however that spectral methods also suffer from similar restrictions on size.

4. Numerical results

The flow region considered was a cubic box in all cases. This permits examination of flows which are initially at a single wavenumber ($\alpha = \pi$), presupposing that the convection cell fills a length of the box. Frequently in the solution the flows altered their wavenumber as time developed. Thus a series of flows were generated for various Rayleigh numbers initially along the 'square cell' wavenumber locus, with particular emphasis on determining the dimensionality of the flow under conditions above each of the two transition points. In each case an initial convective disturbance of a particular wavenumber was introduced into the flow region at time zero, to determine its resilience to instability in favour of other modes. The initial perturbation was given by

$$\Theta = (1 - Y) - f \cos m\pi X \cos l\pi Z, \tag{14}$$

where f is an arbitrary small parameter between ± 1. The flows are summarized in table 1, with comparative descriptions from previous work. The nature of the development and final form of three-dimensional flows are complex and difficult to describe within the confines of two-dimensional diagrams, however a description of each flow in a little detail indicates relevant features.

Cases (f), (d): these two solutions were generated as an initial check on the program to confirm that the three-dimensional algorithm produces results consistent with experiments and with previous two-dimensional numerical studies. It should be emphasized that with an initial (1, 1, 0) modal disturbance introduced into the flow region, there is no possibility of three-dimensional flows unless further stimulation in the third dimension occurs. The algorithm used was sufficiently accurate that in no case was any perturbation due to numerical round-off observed. Such might not always be the case if the system is not as symmetrical as the cube. The Nusselt numbers should therefore be comparable with two-dimensional results (they are in fact consistently about 8 % higher than those quoted throughout this work, but well within the range of experimental results; see for example Gupta & Joseph 1973). No information concerning the stability of the flow to disturbances in the third dimension can be gained. A typical two-dimensional flow is illustrated in figure 1.

Case	R	m	n	l	Observed flow	Aspect ratio	Bories *et al.* (1972) observation	Nu	Predicted 3D Nu	Predicted 2D Nu
(a)	75	1	1	1	Steady 3D	$\sqrt{2}$	3D	2·1	1·78[3]	1·95[3]
(b)	75	1	1	0·8	Steady 3D	$\sqrt{2}$	3D	2·14	1·78[3]	1·95[3]
(c)	75	1	1	0·1	Steady 2D	1	3D	2·25	1·78[3]	1·95[3]
(d)	100	1	1	0	Steady 2D	1	3D	2·8	3·0[2]	2·651[4]
(e)	100	1	1	1	Steady 3D	$\sqrt{2}$	3D	2·9	3·0[2]	2·651[4]
(f)	300	1	1	0	Fluct. 2D	1	Fluct. 2D	~ 4·97	6·5[1]	4·523[4]
(g)	300	1	1	0·5	Steady 3D	$\sqrt{2}$	Fluct. 2D	6·45	6·5[1]	5·016[4]
(h)	300	1	1	0·1	Fluct. 3D	—	Fluct. 2D	6–7	6·5[1]	5·016[4]
(i)	400	1	1	0·1	Fluct. 3D	—	Fluct. 2D	7–8	8·0[1]	6·05[4]

[1] From Gupta & Joseph (1973), upper bound.
[2] From Combarnous & LeFur (1969).
[3] From Zebib & Kassoy (1978), approximate.
[4] From Caltagirone (1974).

TABLE 1

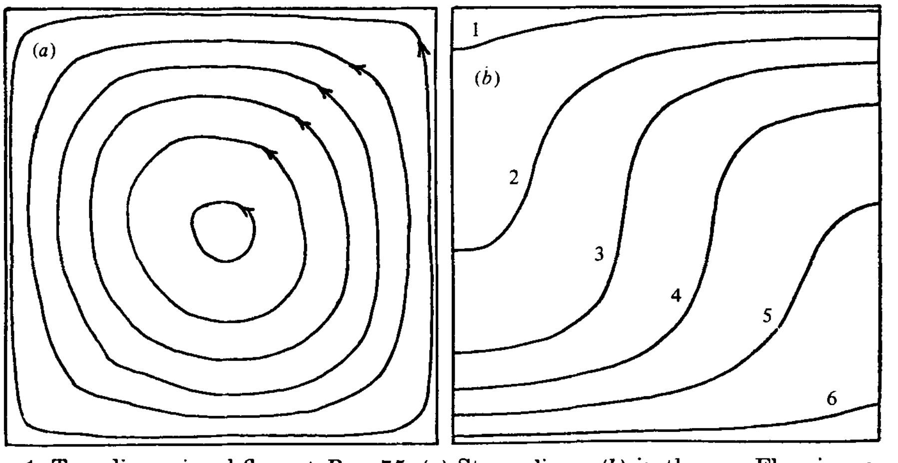

FIGURE 1. Two-dimensional flow at $R = 75$. (a) Streamlines; (b) isotherms. Flow is represented on a vertical X, Y plane. Isotherm numbers 1, 2, 3, 4, 5, 6 correspond to temperatures of $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, respectively.

Cases (*a*), (*e*): these two three-dimensional flows are also falsely determined since there is absolute identity between the two horizontal axes (X and Z), so the flow is constrained to retain its three-dimensional character in the absence of some perturbation with respect to one axis only. The Nusselt numbers obtained give a basis for comparison as to which of the two- or three-dimensional regimes gives rise to greater energy transfer. A typical three-dimensional flow is illustrated in figure 2.

Cases (*b*), (*c*): when the flow initiates in a regime that is almost symmetrically three-dimensional (case *b*), the final mode of flow is three-dimensional and essentially identical to the constrained flow (case *a*). However when the flow is originally almost two-dimensional (case *c*) it becomes completely two-dimensional and follows a roll motion. The paradoxical behaviour of these two flows may be resolved by noting that

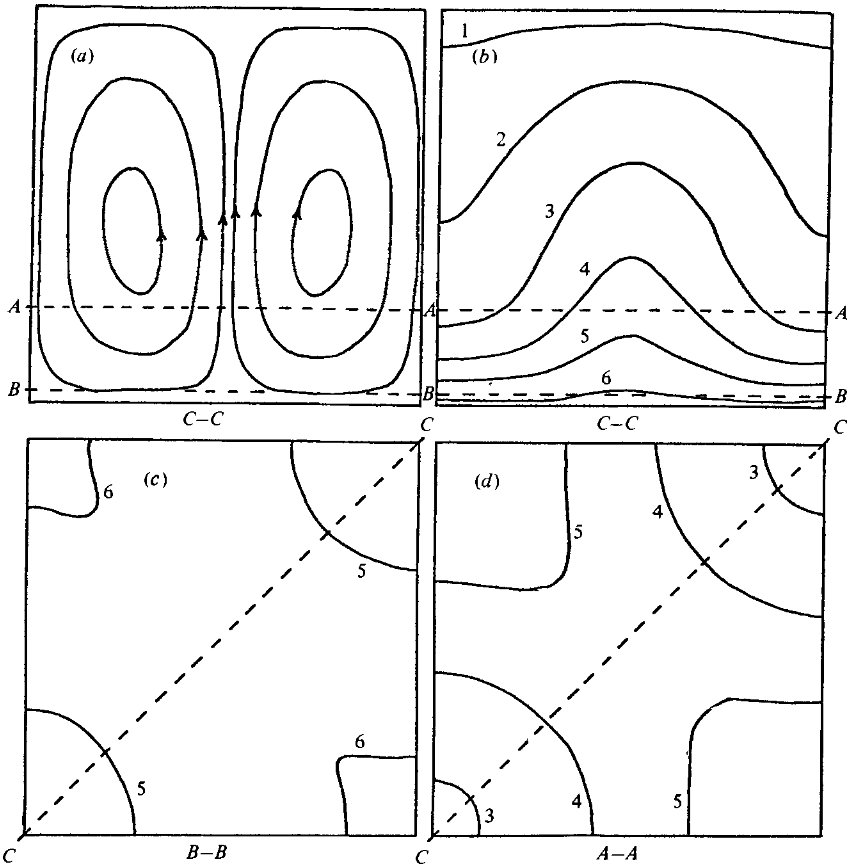

FIGURE 2. Three-dimensional flow at $R = 75$. (*a*) Streamlines on vertical plane diagonal to cube; (*b*) isotherms on vertical plane diagonal to cube; (*c*) isotherms on horizontal plane $Y = \frac{1}{16}$; (*d*) isotherms on horizontal plane $Y = \frac{1}{4}$.

the roll numbers of the final form of the flows are different. The three-dimensional flow consists of two counter-rotating rolls each with axes aligned across the diagonals of the box, hence the wavenumber of each roll is $\sqrt{2}$.

Cases (*g*), (*h*), (*i*): these three flow situations are of particular interest in that they cannot lie within Straus's envelope of stable two-dimensional flows in an infinite layer. Originating from a three-dimensional flow that is dominant in one horizontal dimension (case *g*), the motions tends toward a more symmetrical three-dimensional pattern, and settles down to polyhedral cells of square areal cross-section similar to those described in cases (*b*) and (*c*). These cells may be observed in the isothermal surface plots of figure 3, in which the 0·8, 0·5 and, 0·2 isothermal surfaces are drawn viewed from two different angles. It should be noted that the rising fluid 'ridge' in the 0·8 surface is perpendicular to the falling fluid 'valley' in the 0·2 surface; this observation is characteristic of truly three-dimensional flows, and it is useful to look for this behaviour in more complex flows.

When the flow is originally almost two-dimensional (case (*h*)), the initial motion remains for some time in a two-dimensional convective state (see Horne 1978). However, it later becomes unstable in the third dimension. This flow is somewhat

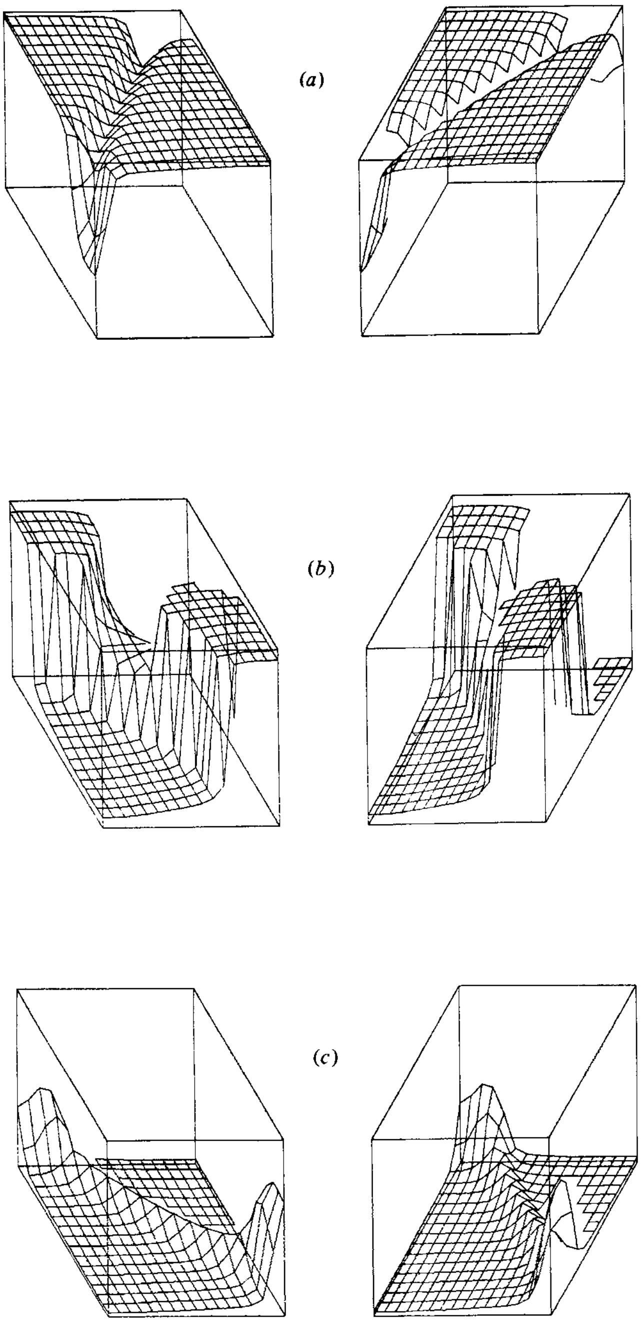

FIGURE 3. Steady three-dimensional flow at $R = 300$. Isothermal surfaces. (a) $\theta = 0{\cdot}2$; (b) $\theta = 0{\cdot}5$; and (c) $\theta = 0{\cdot}8$. Each surface is presented from two viewpoints.

difficult to illustrate, but consists largely of an overall two-dimensional flow with 'waves' moving in the third dimension. This behaviour can be seen in the section diagram sequence in figures 4(*a*)–(*d*) and (*e*)–(*h*), and also in the 0·8 isothermal surface sequence in figures 5(*a*)–(*e*). A 16 mm film of this flow is available, and shows the transient behaviour a little more clearly than is possible here. The 'wave' formation is extremely similar to the evolution of thermal disturbances in the two-dimensional fluctuating convective state. This similarity is even more striking when the Rayleigh number is increased to 400 (case *i*), where two 'waves' can be seen rolling toward one another (figure 6) and finally coalescing and rising as a thermal is formed. The transient behaviour of cases (*h*) and (*i*) can be seen in the plot of Nusselt number against time (figure 7). The time periods are of the order of 0·05 and 0·02 respectively, compared to a time period of around 0·015 for a two-dimensional disturbance at Rayleigh numbers of 400.

5. Conclusions

The behaviour of natural convective flow through porous media in three dimensions is extremely complex, and it is not yet possible to fully describe the overall stability of the flows from the small number of realizations reported here and elsewhere. However, several illuminating facts are evident from this investigation. Consider the initial contention that Bories *et al.* (1972) observed three-dimensional cells in experiments at Rayleigh numbers for which Straus (1974) showed that two-dimensional rolls are stable to a perturbation of any amplitude in the third dimension. It should be remembered that a flow which is stable in the infinite system cannot be unstable in the confined system. It should now be clear that Straus's analysis shows that the two-dimensional rolls *once they exist* are stable to perturbations in the third dimension. The fundamental conclusion of the work by Horne & O'Sullivan (1974) was that more than one flow regime can be observed under identical conditions, depending on the initial conditions. In two dimensions, Horne & O'Sullivan (1974) and Caltagirone (1975) have observed that, as Rayleigh number increases, the wavenumber of the convective disturbance that transfers maximum energy becomes larger. Therefore, it might be expected that even though a flow may start in the form of a unit cell at higher Rayleigh numbers, the wavenumber of the prime disturbances could increase. Such need not be the case, however, since there are situations in which a dominant circulation of lower wavenumber can overcome any attempt by the system to increase in wavenumber, although it is evident from this investigation that this is more likely to occur when the flow is constrained to move in two dimensions than when it is free to move in all three. Following a parallel idea, it might be anticipated that in three dimensions a flow would tend towards either two- or three-dimensional modes depending on which resulted in the greater energy transfer. For example, at Rayleigh number 75, the analysis of Zebib & Kassoy (1978) indicates that a two-dimensional flow will convect more energy than one in three dimensions; this is borne out in these results. However stable flows need not necessarily maximize the energy transfer so the three-dimensional flow observed by Bories *et al.* (1972) is not unexpected, particularly since they are not precluded by the stability analysis of Straus (1974). In view of the more recent stability analysis of Straus & Schubert (1978), it is also interesting to examine the dimensions of the experimental configurations used

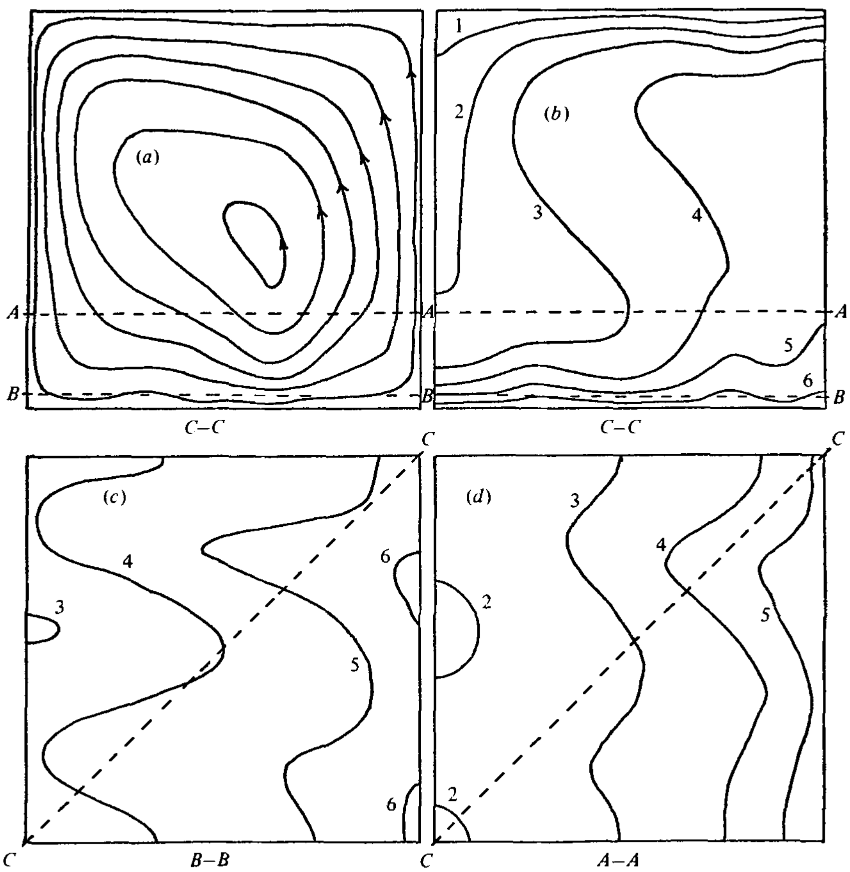

FIGURE 4. For legend see facing page.

earlier. As stated earlier, Bories *et al.* (1972) used a slab-like box 46 × 66 × 5·5 cm deep, and Combarnous & LeFur (1969) used a similar arrangement, 37 × 60 × 5·5 cm deep. These configurations lie deeply in toward the origin in the stability diagram of Straus & Schubert (1978), which considers tall boxes. However, the indications of the analysis by Beck (1972) are that three-dimensional modes are likely with such a wide region, at least close to the linear stability limit. On the other hand, the two-dimensional fluctuating rolls were observed by Caltagirone *et al.* (1971) in a thin vertical slot-like arrangement 38 cm long, 2 cm wide, and 4–6 cm deep. Thus, the two-dimensional nature of this flow is not unexpected.

The behaviour of the unsteady flows observed at Rayleigh numbers of 300 and 400 is also of great interest. The two-dimensional fluctuating convective state was attributed by Horne & O'Sullivan (1978) partly to instability of the thermal boundary layers. Examination of figure 4 reveals that here the instability (at the top and bottom of the box) initiates as a disturbance that causes an overturning (or roll) with its axis perpendicular to the original roll. This is exactly the behaviour proposed by Straus (1974). The mechanism of the fluctuating flow is the same as the two-dimensional case, but the resulting convection is more complex. The properties of the flow at Rayleigh number 300 are also similar to those in the two-dimensional analysis in that both

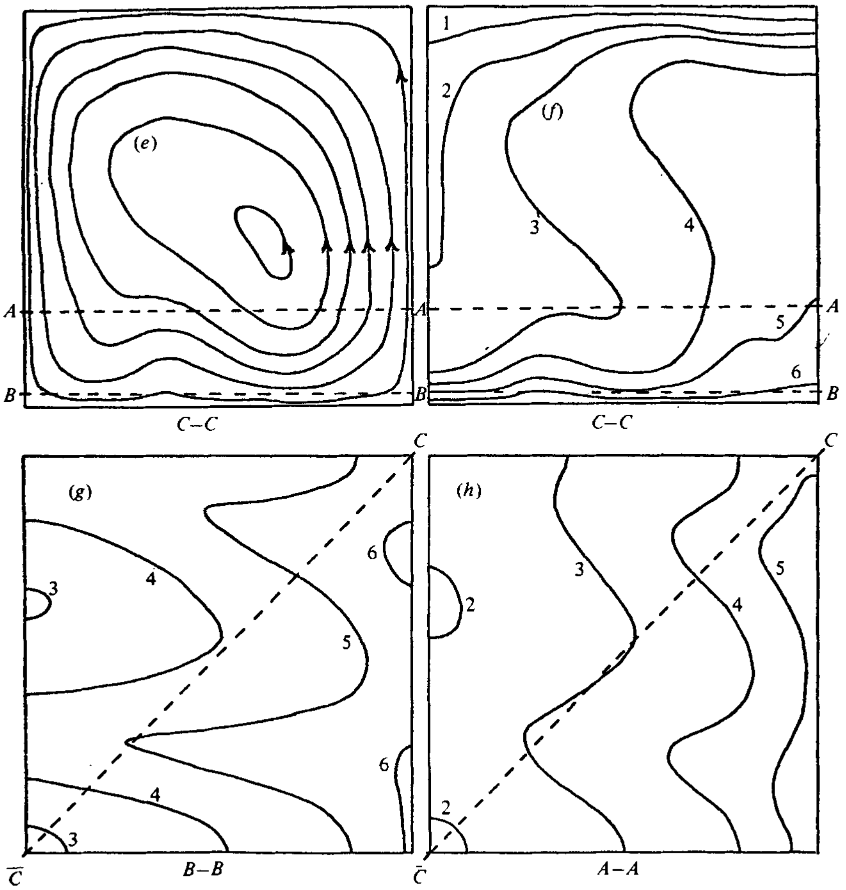

FIGURE 4. (a)–(d) Later three-dimensional flow at $R = 300$. Planes of diagram as in figure 2. (e)–(h) Flow and isothermal pattern, time 0·01 after figure 4(a)–(d).

steady and unsteady alternatives exist. It has already been noted that alternative flow configurations occur at the same Rayleigh number, and Horne & O'Sullivan (1974) have highlighted the influence of confining boundaries on the appearance of one alternative or another.

In conclusion, the natural convective flow of fluid through a porous medium is more complicated than previous studies have shown. It has already been acknowledged that alternative flow regimes exist in purely two-dimensional patterns, and the added freedom in the third dimension increases the number of possibilities. It has therefore been possible to clarify some apparent inconsistencies between previously reported results, which in fact turned out to be not contradictory.

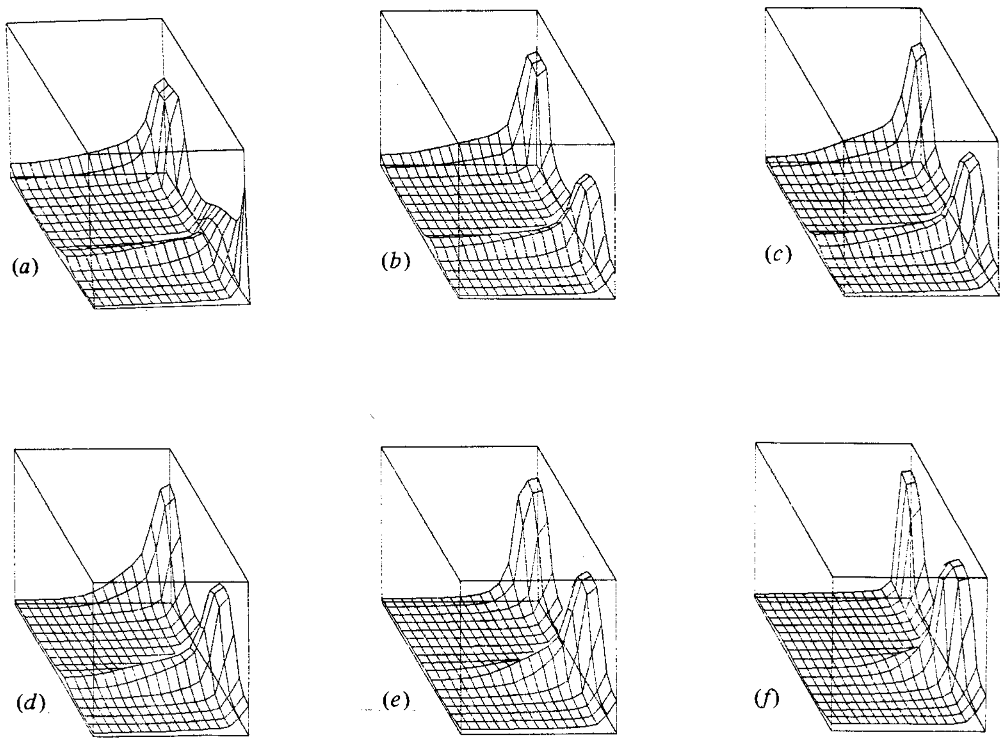

FIGURE 5. Unsteady three-dimensional flow at $R = 300$. Sequence of $\theta = 0{\cdot}8$ isothermal surfaces.

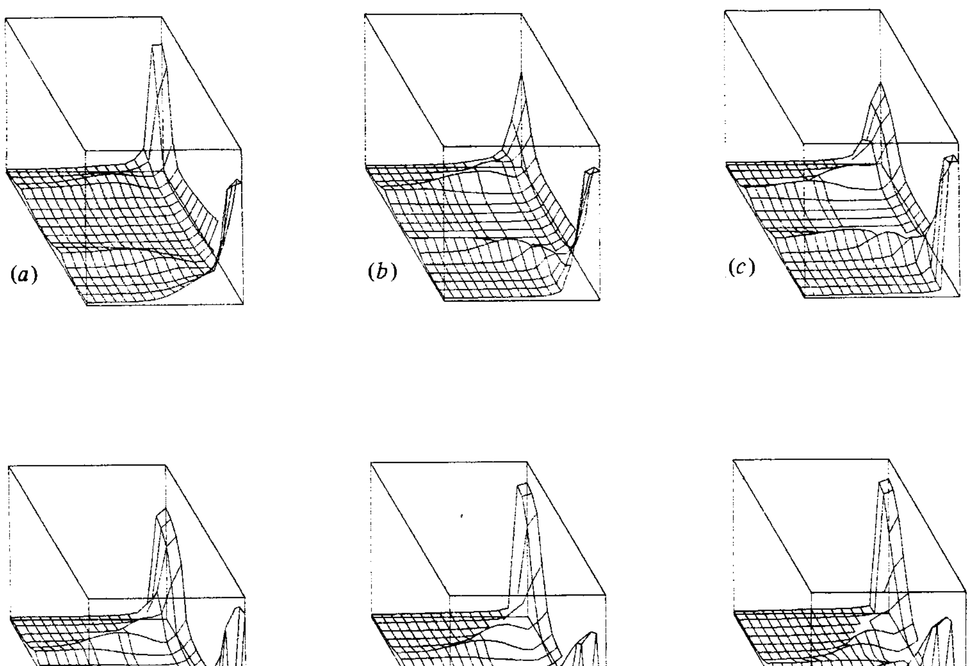

FIGURE 6. Unsteady three-dimensional flow at $R = 400$. Sequence of $\theta = 0{\cdot}8$ isothermal surfaces.

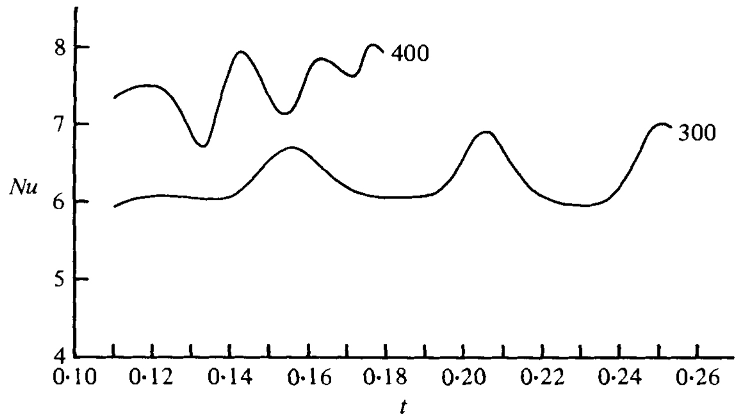

FIGURE 7. Variation of Nusselt number with time for the flows in figures 5 and 6.

This work was made possible by financial assistance from the National Science Foundation Grant NSF (RANN) AER 74-03429-A02, made available by Prof. D. R. Kassoy, of the University of Colorado, and from U.S. Department of Energy Grant DOE EY 76S03-0326PA62, to Prof. G. M. Homsy of Stanford University. The work was performed at Stanford University, Department of Chemical Engineering, while the author was supported by the Stanford Institute of Energy Studies.

REFERENCES

ARAKAWA, A. 1966 *J. Comp. Phys.* **1**, 119.

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

BECK, J. L. 1972 *Phys. Fluids* **15**, 1377.

BORIES, S., COMBRANOUS, M. A. & JAFFRENOU, J. Y. 1972 *Comptes Rendus Acad. Sci. Paris* A **275**, 857.

BUSBEE, B. L., GOLUB, C. H. & NIELSON, C. W. 1970 *SIAM J. Num. Anal.* **7**, 627.

CALTAGIRONE, J. P. 1974 *Comptes Rendus Acad. Sci. Paris* B **278**, 259.

CALTAGIRONE, J. P. 1975 *J. Fluid Mech.* **72**, 269.

CALTAGIRONE, J. P., CLOUPEAU, M. & COMBARNOUS, M. A. 1971 *Comptes Rendus Acad. Sci. Paris* B **273**, 833.

COMBARNOUS, M. A. 1972 *Comptes Rendus Acad. Sci. Paris* A **275**, 1375.

COMBARNOUS, M. A. & LEFUR, B. 1969 *Comptes Rendus Acad. Sci. Paris* B **269**, 1009.

ELDER, J. W. 1967 *J. Fluid Mech.* **27**, 29.

GUPTA, V. P. & JOSEPH, D. D. 1973 *J. Fluid Mech.* **57**, 491.

HIRSH, R. S. 1975 *J. Comp. Phys.* **19**, 90.

HOLST, P. H. & AZIZ, K. 1972 *Int. J. Heat Mass Transfer* **15**, 73.

HORNE, R. N. 1978 Three-dimensional natural convection in a confined porous medium heated from below. Presented at the A.I.A.A.–A.S.M.E. Thermophysics & Heat Transfer Conference, Palo Alto, California, 24–26 May. *Paper A.S.M.E.* 78-HT-56.

HORNE, R. N. & O'SULLIVAN, M. J. 1974 *J. Fluid Mech.* **66**, 339.

HORNE, R. N. & O'SULLIVAN, M. J. 1978 *Phys. Fluids* **21**, 1260.

HORTON, C. W. & ROGERS, F. T. 1945 *J. Appl. Phys.* **16**, 367.

KATTO, Y. & MASUOKA, T. 1967 *Int. J. Heat Mass Transfer* **10**, 297.

LAPWOOD, E. R. 1948 *Proc. Camb. Phil. Soc.* **44**, 508.

ORSZAG, S. A. & ISRAELI, M. 1974 *Ann. Rev. Fluid Mech.* **6**, 281.

PALM, E., WEBER, J. E. & KVERNVOLD, O. 1972 *J. Fluid Mech.* **54**, 153.

PLATZMAN, G. W. 1965 *J. Fluid Mech.* **23**, 481.

RUBIN, H. 1974 *J. Hydrology* **21**, 173.

SCHUBERT, G. & STRAUS, J. M. 1978 Three-dimensional and multi-cellular steady and unsteady convection in fluid-saturated porous media at high Rayleigh numbers. *J. Fluid Mech.* To be published.

STRAUS, J. M. 1974 *J. Fluid Mech.* **64**, 51.

STRAUS, J. M. & SCHUBERT, G. 1977 *J. Geophys. Res.* **82**, 325.

STRAUS, J. M. & SCHUBERT, G. 1978 *J. Fluid Mech.* **87**, 385.

YIH, C. S. 1969 *Fluid Mechanics*. New York: McGraw-Hill.

ZEBIB, A. & KASSOY, D. R. 1978 *Phys. Fluids* **21**, 1.